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1988 J. Phys. A: Math. Gen. 21 4609

(http://iopscience.iop.org/0305-4470/21/24/015)

J. Phys. A: Math. Gen. 21 (1988) 4609–4616. Printed in the UK

Looking at the equilibrium measures in dynamical systems

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Received 1 June 1988

Abstract. We discuss how finite-time fluctuations of the chaoticity degree permit us to observe a family of equilibrium measures defined in the thermodynamical formalism for expanding one-dimensional maps and for axiom A systems. By means of generalised Lyapunov exponents one can thus calculate the Kolmogorov entropies, the Lyapunov exponents and the Hausdorff dimensions for this set of measures. We perform such a calculation for the Lozi map, an almost everywhere hyperbolic diffeomorphism of the plane. We stress the heuristic power of our approach, which can be extended to more generic non-hyperbolic systems. In this case we suggest that phase transition phenomena might appear as a consequence of the existence of 'laminar-like' regular periods during chaotic evolutions.

Temporal intermittency is a typical feature of chaotic systems and is usually measured by means of the generalised Lyapunov exponents, $L(q)$, and of the Renyi entropies, K_q (see, e.g., Paladin and Vulpiani 1987 and references therein). It has recently been proposed to regard intermittency as a multifractal object in trajectory space (Eckmann and Procaccia 1986, Paladin *et al* 1986, Szepfalusy and Tel 1987), by linking the Renyi entropies to the scaling of the probability distribution that rules the finite-time fluctuations of the chaoticity degree via a Legendre transformation. This approach is quite analogous to that used for characterising the singularity structure of a probability measure on strange attractors (Benzi *et al* 1984, Halsey *et al* 1986). Our purpose is to make clear the connection of these approaches to the thermodynamic formalism by introducing a family of invariant ergodic measures which are observable on finite times even if, asymptotically, just one of them describes the global features of the system. It is thus possible to obtain the analogue of the partition function in the statistical mechanics formalism (Ruelle 1978, Walters 1978) by estimating the finite-time fluctuations. Actually, it is more convenient to reconstruct the probability distribution which rules them via a calculation of the moments of suitable observables.

Let us consider the fluctuations of the chaoticity degree by introducing the finite-time Lyapunov exponents of a flow or of a map $F^t(x)$ according to whether t is a continuous variable or an integer:

$$\gamma(x, t) = (1/t)\ln|D_x F^t(x)| \tag{1}$$

where x belongs to the F-invariant set $J \subset \mathbb{R}^d$ and $|D_x F^t(x)|$ is the norm of the tangent map of the transformation F^t at x. It is clearly evident that a small error δx in our knowledge of x grows exponentially as $|\delta x(t)| \propto |\delta x|\,e^{\gamma t}$ where $x(t) = F^t(x)$.

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In order to measure a global chaoticity degree, one usually considers the maximum Lyapunov characteristic exponent, LCE:

$$\lambda(\mu)=\lim_{t\to\infty}\frac{1}{t}\int_J \ln|D_xF^t|\,\mathrm{d}\mu(x)\equiv\lim_{t\to\infty}\langle\gamma(x,t)\rangle_\mu$$
$$=\lim_{t\to\infty}\gamma(x,t)\qquad\text{for } \mu \text{ negative for almost all } x \tag{2}$$

where μ is an ergodic invariant measure and the compact set J is its support. In the following we shall assume that the dynamics individuates a 'physical measure' ρ to which all experimental measures refer. Now, $\gamma(x, t)$ fluctuates at varying x around its average value $\langle\gamma(x, t)\rangle_\rho$, ρ negative almost everywhere, but for large enough times we can take $\langle\gamma(x, t)\rangle_\rho=\lambda(\rho)$ for practical purposes. Indeed, typical corrections to the asymptotic limit λ are $O(1/t)$ (Bouchaud *et al* 1988) and can be neglected with respect to the amplitude of the large fluctuations which are expected to be $O(1/t^{1/2})$ in most cases (Goldirsch *et al* 1987).

In order to reconstruct the scaling properties of the probability $\mathscr{P}(\gamma)$ of measuring a value of the finite-time LCE in the interval $[\gamma, \gamma+\mathrm{d}\gamma]$, when the initial point x is chosen ρ negative almost everywhere, let us define the generalised Lyapunov exponents $L(q)$ which are asymptotic indices, quite easily computed in numerical calculations. They are the moments of a small perturbation on the trajectory of dynamical systems:

$$L(q)=\lim_{t\to\infty}(1/t)\ln\langle|D_xF^t|^q\rangle_\rho=\lim_{t\to\infty}(1/t)\ln\int\mathscr{P}(\gamma)\,\mathrm{d}\gamma\exp(\gamma qt). \tag{3}$$

We have to assume an exponential decay of $\mathscr{P}(\gamma)$, i.e.

$$\mathscr{P}(\gamma)\propto\exp(-\psi(\gamma)t) \tag{4}$$

if we want to obtain a finite $L(q)$ in (3). Note that $\mathscr{P}(\gamma)$, $L(q)$ and $\psi(\gamma)$ all refer to the natural measure ρ if there are no other explicit labels. A saddle-point estimate of (3) gives the Legendre transformation of the function $\psi(\gamma)$:

$$L(q)=\max_\gamma(q\gamma-\psi(\gamma))=\bar{\gamma}q-\psi(\bar{\gamma}) \tag{5}$$

where

$$\bar{\gamma}(q)=\mathrm{d}L/\mathrm{d}q. \tag{6}$$

$\bar{\gamma}$ can vary in the interval $[\gamma_{\min}, \gamma_{\max}]$ and $\gamma_{\min}$ ($\gamma_{\max}$) is the minimum (maximum) γ value selected by $q\to-\infty$ ($q\to\infty$), while for $q\to0$ one sees that $\mathrm{d}L/\mathrm{d}q=\lambda$. From general theorems of probability theory on the moments, one can show that $L(q)$ is a convex function of q, i.e. $\bar{\gamma}(q)$ is a non-decreasing function of q. In the limit of small q (dominated by the most probable fluctuations), one can stop the Taylor expansion of $L(q)$ at the second order: $L(q)=\lambda q+(b/2)q^2$. This means that for values of γ close to the Lyapunov exponent value the γ distribution is well approximated by the normal distribution

$$\psi(\gamma)=(\gamma-\langle\gamma\rangle)^2/2b. \tag{7a}$$

Such a distribution can be obtained by regarding the γ fluctuations as a random process:

$$\gamma(x,t)=\langle\gamma\rangle_\rho+\xi(x)t^{-1/2} \tag{7b}$$

where ξ is a random variable of zero mean value and variance b. However, a second-order approximation for $L(q)$ is generally not satisfactory for $q\gtrsim1$.

The main point of this paper is that a statistical mechanics approach can be achieved by numerical calculations of the generalised Lyapunov exponents. Our strategy is as follows.

(i) We show that the maximum Lyapunov exponents $\lambda(\mu_\beta)$, the Kolmogorov entropies $h(\mu_\beta)$ and the Hausdorff dimensions $\mathrm{HD}(\mu_\beta)$ related to the set of equilibrium measures $\{\mu_\beta\}$ of the function $-\beta u(x) = -\beta \ln|D_x F|$ with $\beta \in \mathbb{R}$ for uniformly expanding maps of the interval can be derived by the $L(q)$. Most of our results can be extended to axiom A (Bowen 1973) attractors provided that $|D_x F|$ is replaced by the restriction of the tangent map along the unstable direction†. Let us recall that axiom A systems are hyperbolic and so the finite-time maximum Lyapunov exponents cannot vanish, i.e. $\gamma(x, t) > 0$ for t large enough and $x \in J$, as well as in expanding maps. In these cases μ_β is the unique ergodic measure which realises the supremum in the variational principle (Walters 1975, Ruelle 1978, 1982) for the topological pressure $P(\beta)$:

$$P(\beta) = \sup_{\mu \in M_F(J)} \left(h(\mu_\beta) - \beta \int u(x)\, d\mu(x) \right) = h(\mu_\beta) - \beta\lambda(\mu_\beta) \tag{8}$$

where $M_F(J)$ is the set of the F-invariant measure on the F-invariant set J, $h(\mu)$ is the Kolmogorov entropy of μ and $\lambda(\mu) = \int d\mu(x) u(x)$ (see, e.g., Eckmann and Ruelle 1985).

(ii) We explicitly give $h(\mu_\beta)$, $\lambda(\mu_\beta)$ and $\mathrm{HD}(\mu_\beta)$ via a numerical calculation of the generalised Lyapunov exponents $L(q)$ in the Lozi map (see Collet and Levy (1984) for some rigorous results on this system) a two-dimensional quasihyperbolic diffeomorphism.

(iii) We discuss how a phase transition (jumps in the entropy at a certain critical value of β) can appear in dynamical systems which are not axiom A as a consequence of the existence of regular 'laminar-like' periods during chaotic evolution.

To obtain (i) we need to use the key relation which links the topological pressure to the generalised Lyapunov exponents $L_\beta(q)$ obtained by an ensemble average taken over μ_β (Bessis *et al* 1988):

$$L_\beta(q) \equiv \lim_{t\to\infty} (1/t) \ln\langle|D_x F^t|^q\rangle_{\mu_\beta} = P(\beta - q) - P(\beta). \tag{9}$$

One can show (Walters 1978) as a particular case that

$$\lambda_\beta \equiv \lambda(\mu_\beta) = -dP/d\beta. \tag{10}$$

By comparing (10) and (9) with (6), one has a direct interpretation of the derivative of the generalised Lyapunov exponent:

$$\bar{\gamma}_\beta(q) = dL_\beta/dq = -\left.\frac{dP(z)}{dz}\right|_{z=\beta-q} = \lambda_{\beta-q}. \tag{11}$$

This relation makes it evident that knowledge of finite-time fluctuations via generalised Lyapunov exponents with respect to only *one* equilibrium measure is fully equivalent to knowledge of the LCE computed with respect to all the equilibrium measures $\{\mu_\beta\}$. Formulae (9)-(11) can be rigorously proven for a uniformly expanding map of the

† In contrast to uniformly expanding maps of the interval, for two-dimensional axiom A maps, one has to consider a finite-time 'energy density' $u_t = -(1/t) \ln|D_x F^t|$ and then to extrapolate the thermodynamic limit of the 'energy' function $U_t = \int u_t(x)\, d\mu(x)$. This is often simple since the corrections are of the form $U_t = U_\infty + C/t$ in most systems.

interval. However they remain valid for the Baker transformation and they are conjectured to hold with respect to the Bowen-Ruelle-Sinai measure for which $\beta = 1$ and $P(1) = 0$ in the case of two-dimensional axiom A attractors when the pressure is computed along the unstable direction. We now extend this conjecture to all the equilibrium measures $\{\mu_\beta\}$ with $\beta \in \mathbb{R}$.

We thus see that the Legendre transformation (5) and the variational principle (8) imply that the probability of finding a finite-time fluctuation $\gamma(x, t)$ is related to the Kolmogorov entropy of a particular equilibrium measure:

$$h(\mu_{\beta-q}) = \beta\bar{\gamma}_\beta - \psi(\bar{\gamma}_\beta) + P(\beta) \tag{12}$$

where $\psi(\gamma_\beta)$ and γ_β are related to $L_\beta(q)$ by (5) and (11).

Now we go into a little more detail about the statistical properties of the probability function $\mathcal{P}(\gamma)$ but, to do this, we must introduce some technical definitions. First, we relate the following considerations to uniformly hyperbolic one-dimensional systems, but the same considerations can be extended to axiom A attractors considering the dynamics 'projected' along the unstable directions as we have already pointed out. It is well known that, for the expanding systems we are considering, there is a finite partition of the invariant set J into sufficiently small closed sets, called cylinders or rectangles, such that the partition $F^{-t}\mathcal{A}$ is a refinement of $F^{-t+1}\mathcal{A}$ and diam $F^t\mathcal{A} \to 0$ when $t \to -\infty$ (the Markov partition). Following Bohr and Rand (1987) we introduce the cardinality $N_t(\gamma)$ of the set R_t of rectangles belonging to $F^{-t}\mathcal{A}$ so that the finite-time Lyapunov exponent γ, starting from some point in R_t, belongs to the interval $I = [\bar{\gamma}, \bar{\gamma} + \mathrm{d}\gamma]$. Roughly speaking, $N_t(\bar{\gamma})$ is the number of trajectories with a finite-time LCE $\bar{\gamma}$. For large t, $N_t(\gamma)$ scales as (Bohr and Rand 1987)

$$N_t(\gamma) \propto \exp(S(\gamma)t). \tag{13}$$

$S(\gamma)$ is here the topological entropy (Bowen 1973) of the set of points for which $\gamma(x, t)$ converges to $\bar{\gamma}$. It follows that $S(\gamma) \leqslant h_{\mathrm{top}}$ where

$$h_{\mathrm{top}} = \sup_{\mu \in M_{\mathrm{F}}(J)} h(\mu) = h(\mu_{\beta=0}) = P(0) \tag{14}$$

is the topological entropy of the invariant set J. It is indeed simple to establish a relation between $S(\gamma)$ and $\psi(\gamma)$ defined by (3). Let us in fact recall that $\mathcal{P}(\bar{\gamma}_\beta)$ is the probability of finding $\gamma(x, t)$ in the interval $I = [\bar{\gamma}_\beta, \bar{\gamma}_\beta + \mathrm{d}\gamma]$ when the initial point is chosen μ_β negative almost everywhere. We thus get

$$\mathcal{P}(\gamma_\beta) = \mu_\beta(R_t) \tag{15}$$

where the set R_t, which gives the 'good' finite-time LCE, is weighted by the measure μ_β.

Now, if A^t_α is an element of $R_t \in F^{-t}\mathcal{A}$, by applying the theory of Walters (1978) for the Perron-Frobenius operator it is possible to show (Vaienti 1988) that

$$\mu_\beta(A^t_\alpha) \propto \exp[t(\varepsilon - P(\beta))]|D_x F^t|^{-\beta}$$

where x is any point in A^t_α and ε is an arbitrary positive constant independent of x and t. Choosing x, for each A^t_α, in the set of points which gives $\gamma(x, t) \in I$ and sending $\varepsilon \to 0$, we finally have

$$\mathcal{P}(\bar{\gamma}_\beta) = \mu_\beta(R_t) = N_t(\bar{\gamma}_\beta)\exp[-t(P(\beta) + \beta\bar{\gamma}_\beta)]$$

and comparison with (16) and (15) gives the relation

$$\psi(\bar{\gamma}_\beta) = \beta\bar{\gamma}_\beta - S(\bar{\gamma}_\beta) + P(\beta). \tag{16}$$

Let us emphasise the thermodynamical analogy. If we identify $\bar{\gamma}_\beta$ with the energy, $S(\bar{\gamma}_\beta)$ with the microcanonical entropy and $-P(\beta)/\beta$ with the free energy, the probability of finding an energy per particle $\bar{\gamma}_\beta$ different from the average value λ_β in a system with N particles decays as $\exp(-N\psi(\gamma_\beta))$. One sees that $S(\bar{\gamma}_\beta)$ does not depend on the particular equilibrium measure chosen, but the parametrisation $\bar{\gamma}_\beta(q)$ depends on it, via (11). Moreover, from (10), it is easy to obtain

$$S(\bar{\gamma}_\beta(q)) = h(\mu_{\beta-q}) \tag{17}$$

where $\bar{\gamma}_\beta(q) = \mathrm{d}L_\beta/\mathrm{d}q$. In the limit $q \to 0$, it becomes

$$S(\lambda_\beta) = h(\mu_\beta) \tag{18}$$

which has been derived by Bohr and Rand (1987). Equation (18) is trivial to understand by physical intuition since it corresponds to the existence of a 'thermodynamic limit' for $\gamma(x, t)$, with respect to the measure μ_β as stated by the Oseledec theorem (Oseledec 1968). Indeed, one sees that (18) is equivalent to assuming $\psi(\lambda_\beta) = 0$ in (16) so that the probability of finding $\gamma(x, t) = \lambda_\beta$ does not vanish when $t \to \infty$.

On the contrary, (17) has an 'experimental' relevance. In fact for axiom A attractors the physical measure on the attractor is smooth along the unstable manifold (the Sinai-Bowen-Ruelle measure) so that we get the topological pressure function as $P(x) = L(q = 1 - x)$ since $P(1) = 0$ by extending the Bowen-Ruelle relation (McCluskey and Manning 1983). It follows that Lyapunov exponents and Kolmogorov entropies related to averages taken over the uncountable set of equilibrium measures μ_β are in terms of $L(q)$:

$$\begin{aligned} \lambda(\mu_{\beta=1-q}) &= \bar{\gamma} = \mathrm{d}L/\mathrm{d}q \\ h(\mu_{\beta=1-q}) &= S(\bar{\gamma}) = \bar{\gamma} - \psi(\bar{\gamma}) = (1-q)\,\mathrm{d}L/\mathrm{d}q + L(q). \end{aligned} \tag{19}$$

In figure 1 we show $\bar{\gamma}(q)$ and $\psi(\bar{\gamma})$ and in figure 2 the corresponding $h(\mu_\beta)$ and $\lambda(\mu_\beta)$ as a function of β for the Lozi map. Note that $\lambda(\mu_\beta)$ is a non-increasing function of β since $L(q)$ is a convex function, while $h(\mu_\beta)$ increases for negative β and decreases for positive β; its maximum value is the topological entropy $h_{\mathrm{top}} = L(1)$, corresponding to the so-called maximum entropy measure selected by $\beta = 0$.

It is worth stressing that for two-dimensional axiom A attractors we can also compute the Hausdorff dimensions $\mathrm{HD}(\mu_\beta)$ of the measures μ_β (often called information dimensions) by the relation (Young 1982)

$$\mathrm{HD}(\mu) = h(\mu)\left(\frac{1}{\lambda(\mu)} - \frac{1}{\chi_-(\mu)}\right) \tag{20}$$

where χ_- denotes the smallest Lyapunov exponent. For maps with constant Jacobian j, where $\chi_- = \ln j - \lambda$, (20) can be written as

$$\mathrm{HD}(\mu_{\beta=1-q}) = \left(1 - \frac{\psi(\bar{\gamma}(q))}{\bar{\gamma}(q)}\right)\left(1 + \frac{\bar{\gamma}(q)}{|\ln j| + \bar{\gamma}(q)}\right) \tag{21}$$

where $\bar{\gamma}(q)$ and $\psi(\bar{\gamma}(q))$ are given by (6) and (5) respectively. For the physical measure (i.e. for $q \to 0$) (20) reduces to the Kaplan and Yorke formula (Kaplan and Yorke 1978). Roughly speaking, $\mathrm{HD}(\mu)$ is the Hausdorff dimension of the smallest subset of the attractor of full μ measure. It can be proved (McCluskey and Manning 1983) that generically there are no equilibrium measures for which $\mathrm{HD}(\mu) = d_{\mathrm{H}}$, the Hausdorff dimension of the attractors and $\sup_\mu \mathrm{HD}(\mu) < d_{\mathrm{H}}$.

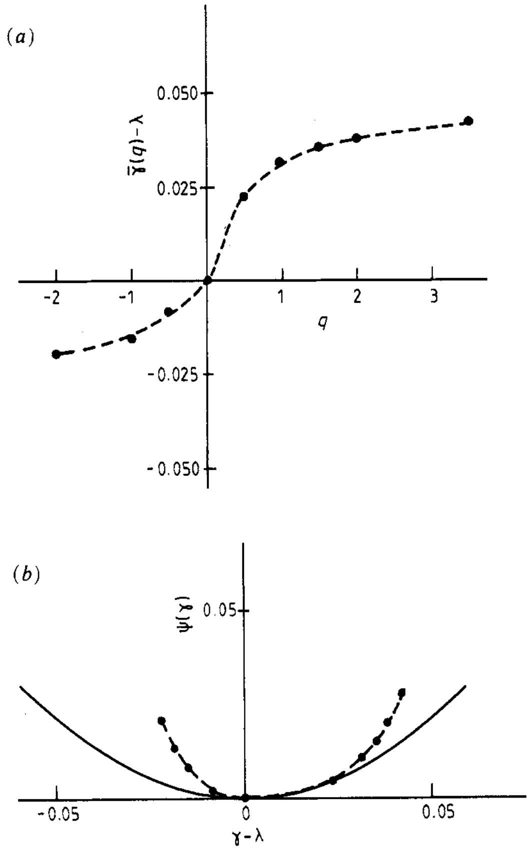

Figure 1. (*a*) $\bar{\gamma}-\lambda$ against q for the Lozi map $x_{n+1}=a|x_n|+y_n+1$, $y_{n+1}=bx_n$, with $a=1.7$ and $b=0.5$; $\lambda=0.470$. The points are obtained by a numerical calculation of $L(q)$ via formula (6). (*b*) $\psi(\gamma)$ against $\gamma-\lambda$ for the Lozi map. Points are obtained by the Legendre transformation (5) of the generalised Lyapunov exponents. The full curve indicates the normal approximation (7*a*).

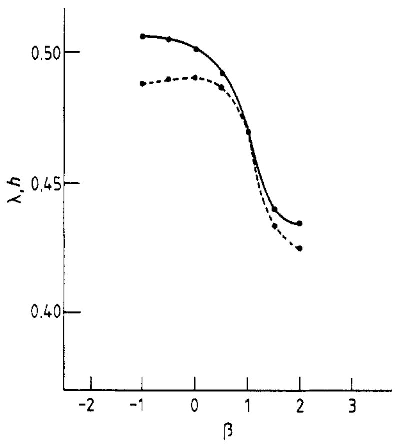

Figure 2. $h(\mu_\beta)$ (broken curve) and $\lambda(\mu_\beta)$ (full curve) against β for the Lozi map from the numerical results shown in figure 1.

Figure 3 shows HD(μ_β) computed by (21) in the Lozi map. The maximum of HD(μ_β) seems to be reached by β values close to 1, and is smaller than the Hausdorff dimension of the attractor, estimated to be $d_H = 1.415 \pm 0.005$.

The heuristic power of our approach is evident. It makes clear in what sense the equilibrium measures are observable in the set of the invariant ergodic measures. Moreover one can try to extend it to more generic systems which are not of the axiom *A* type. Let us consider as a prototype a simple two-dimensional non-hyperbolic map, the Henon map. In this case the condition $\gamma_{min} > 0$ does not hold in a set of 'turn back' points (homoclinic tangencies) where the orbit becomes marginally stable (Gunaratne and Procaccia 1987). This suggests that for negative $q < q_c$, which pick up low 'temperatures' β^{-1} via (19), an ordered phase can appear corresponding to the possibility of finding the system in a laminar regular state characterised by a finite-time LCE $\bar{\gamma}(q)$ which is non-positive. It is an open question what are the probability measures which support such non-positive LCE, since they cannot be indecomposable, i.e. ergodic. One can conjecture that the laminar regions tested by negative q are equivalent to the different pure phases of spin systems below a transition temperature. A first-order phase transition should appear as an edge in the pressure at $\beta_c = 1 - q_c$ and so in the function $L(q)$. Unfortunately this is rather difficult to detect by a numerical calculation of $L(q)$. However, Procaccia and co-workers (Gunaratne and Procaccia 1987, Jensen 1988) provided evidence of a jump of an entropy-like function, the so-called $f(\alpha)$ spectrum (Halsey *et al* 1986), in the Henon map due to the existence of the set of non-hyperbolic 'turnback' points.

Let us finally remark that in most papers the finite-time fluctuations of the Kolmogorov entropy are considered instead of those of the LCE. In this case, one measures the Renyi entropies K_q (see, e.g., Eckmann and Ruelle 1985, Paladin and Vulpiani 1987) where $K_1 = h(\mu)$ and $K_0 = h_{top}$. A Legendre transformation, quite analogous to (5), relates the set of K_q to a function corresponding to $S(\gamma)$ (Eckmann and Procaccia 1986, Paladin *et al* 1986, Szepafalusy and Tel 1987).

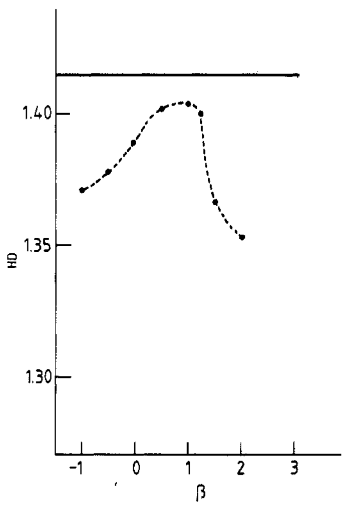

Figure 3. HD(μ_β) against β obtained by (21) for the Lozi map. The horizontal full line indicates the value of the Hausdorff dimension of the attractor; the information dimension of the physical measure is HD($\mu_{\beta=1}$) = 1.404.

The link with the variational principle is, however, less transparent since for expanding maps of the interval one has the more involved relation (Bessis *et al* 1988):

$$(q-1)K_q(\mu_\beta) = qP(\beta) - P(\beta q). \tag{22}$$

Nevertheless, for the physical measure, i.e. for $\beta = 1$, (22) and (9) imply

$$K_{1-q} = P(1-q)/q = L(q)/q \tag{23}$$

for hyperbolic maps of the plane and so the equivalence of the two different sets of exponents. Equation (23) can be also extended to higher-dimensional systems (Paladin and Vulpiani 1986).

Let us conclude by the reasonable suggestion that our arguments are 'generic' in the sense that, for non-hyperbolic maps of the plane, (21) as well as most of our results remain valid for q values less than the critical q value corresponding to the phase transition in the topological pressure (Grassberger *et al* 1988).

Acknowledgments

We have enjoyed an interesting correspondence with T Tel. One of us (GP) wants to thank M H Jensen, A Porzio and I Procaccia for useful discussions and suggestions. SV acknowledges the support of a CNR-NATO fellowship.

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